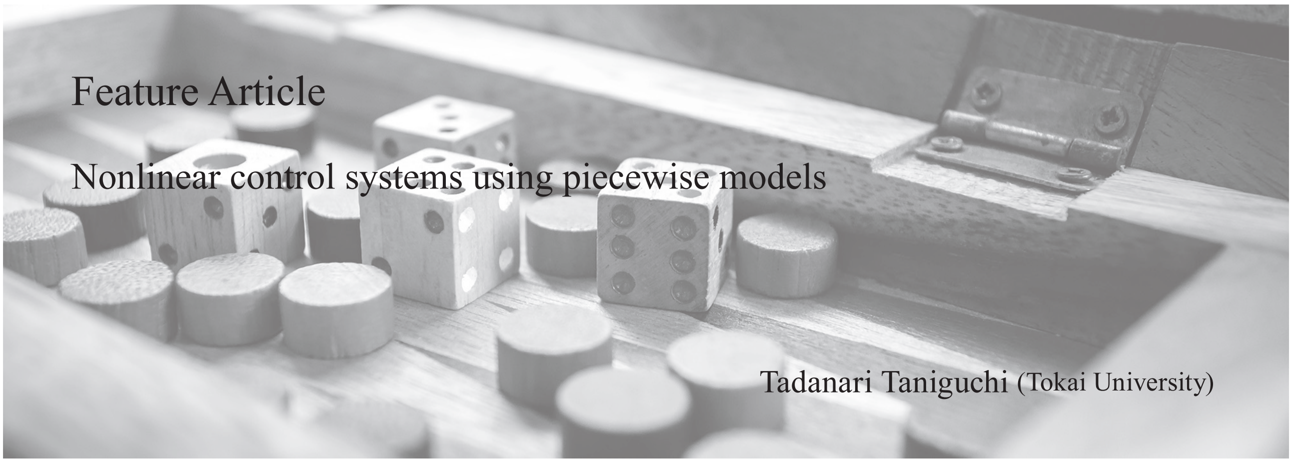

Feature Article

Nonlinear control systems using piecewise models

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1. Piecewise model approaches for nonlinear controls

Despite the many analysis and synthesis methods proposed for nonlinear control systems, difficulties remain in dealing with nonlinear control systems, compared to linear systems, and the major reason lies in the lack of a general parametric expression for nonlinear systems, whereas linear systems are parametric by nature and hence operational.

We proposed Piecewise Multi-Linear (PML) modeling and Input-Output (I/O) linearization as a powerful combination [Taniguchi and Sugeno, 2013] for analyzing and synthesizing nonlinear control systems, where the PML system was parametric like the linear system and I/O linearization. Feedback linearization is a generalmethod for dealingwith nonlinear control systems despite limitations in applications.

Piecewise Linear (PL) systems have been intensively studied in connection with nonlinear systems [Johansson and Rantzer, 1998], The original idea was to parametrically approximate a nonlinear function with PL functions [Babayev, 1997]. An important class of hybrid systems is PL systems with a set of rules for switching among systems [Imura and van der Schaft, 2000], where state space is divided into polyhedral or polytopic regions, each region associated with a linear (or affine) system. Gain scheduling is also considered with the PL approach [Shamma and Athans, 1990]. The PL system concept appears in TS systems [Tanaka and Wang, 2001] that approximate general nonlinear systems with a number of rules, but unlike the conventional PL approximation, these systems are not fully parametric.

This work concerns parametric piecewise approximation of nonlinear control systems based on the original idea of PL approximation. PL approximation has a general approximation capability for nonlinear functions with given precision, but the PL system obtained is too complex to use for control purposes. To overcome this difficulty, it has been suggested to use PML approximation [Sugeno, 1999]. The PML model has the following features:

- It is built on hyper-cubes partitioned in the state space.
- It has general approximation capability for nonlinear systems.
- It is a piecewise nonlinear model, the second simplest after the PL model.
- It is continuous and fully parametric.

We have thus far shown the necessary and sufficient conditions for PML system stability for Lyapunov functions in the twodimensional case [Sugeno and Taniguchi, 2004; Taniguchi and Sugeno, 2004]. Because stabilizing conditions are represented by Bilinear Matrix Inequalities (BMIs) [Goh et al., 1994], however, long computing time to obtain a stabilizing controller.

To overcome these difficulties, we have derived stabilizing conditions [Taniguchi and Sugeno, 2010a; 2010b; 2012a] based on feedback linearization, where [Taniguchi and Sugeno, 2010a] and [Taniguchi and Sugeno, 2012a] apply input-output linearization and [Taniguchi and Sugeno, 2010b] applies full-state linearization. In feedback linearization, we design a state feedback controller that transforms a nonlinear system into an equivalent linear system. Feedback linearization is a very powerful tool for synthesizing nonlinear control systems, but it is not always applicable because of strict linearization conditions, i.e., the linearizable region is often local.

For this reason, the last three decades have been spent studying approximate linearization via feedback. Approximate linearization was proposed in the literature based on four streams [Taniguchi and Sugeno, 2012a]: partial linearization, linearization-oriented modeling, nonlinearity measures, and linear model matching.

This article deals with PML system I/O linearization along the line of approximate linearization, demonstrating that if nonlinear systems are modeled with PML systems, they become easily and globally feedback linearizable with Look-Up-Table (LUT) controllers. LUT controllers are widely used in industrial control, specifically, for vehicle control because of simplicity and visibility. PML modeling gives rise to approximation error. We proposed a method [Taniguchi and Sugeno, 2012b] for robustly stabilizing PML systems, and approximation error in modeling is considered as discussed in [16]. These control systems have the following features: (1) Researchers need only partial knowledge of vertices in piecewise regions, not overall knowledge of an objective plant. (2) These control systems are applicable to a wider class of nonlinear systems than conventional I/O linearization. (3) The PML model and controller are represented as an LUT.

This article is organized as follows. Section II presents the canonical form of PML models. Section III proposes the design of LUT controllers for nonlinear plants based on PML models and I/O linearization. Section IV presents examples demonstrating the feasibility of the proposed methods. Section V summarizes conclusions.

2. Canonical form of piecewise bilinear models

2.1 Open-loop systems

In this section, we introduce the PML models suggested in [Sugeno, 1999]. We deal with the two dimensional case without loss of generality.

We assume that w_1^σ and w_2^τ are normalized membership functions of a triangular form. Figure 1 shows the normalized triangular membership function $w_1^\sigma(x_1)$. Define a vector $d(\sigma, \tau)$ and a rectangle $R_{\sigma\tau}$ in the two-dimensional space as, respectively,

$$d(\sigma, \tau) \equiv (d_1(\sigma), d_2(\tau))^T,$$
$$R\sigma\tau \equiv [d_1(\sigma), d_1(\sigma + 1)] \times [d_2(\tau), d_2(\tau + 1)]. \quad (1)$$

σ and τ are integers: $-\infty < \sigma, \tau < \infty$ where $d_1(\sigma) < d_1(\sigma + 1)$, $d_2(\tau) < d_2(\tau + 1)$ and $d(0, 0) \equiv (d_1(0), d_2(0))^T$. The superscript ^{T} denotes transpose operation.

For $x \in R\sigma\tau$, the PML system is expressed as

$$\begin{cases} \dot{x} = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} w^i{}_1(x_1) w^j{}_2(x_2) f_o(i, j), \\ x = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} w^i{}_1(x_1) w^j{}_2(x_2) d(i, j). \end{cases} \quad (2)$$

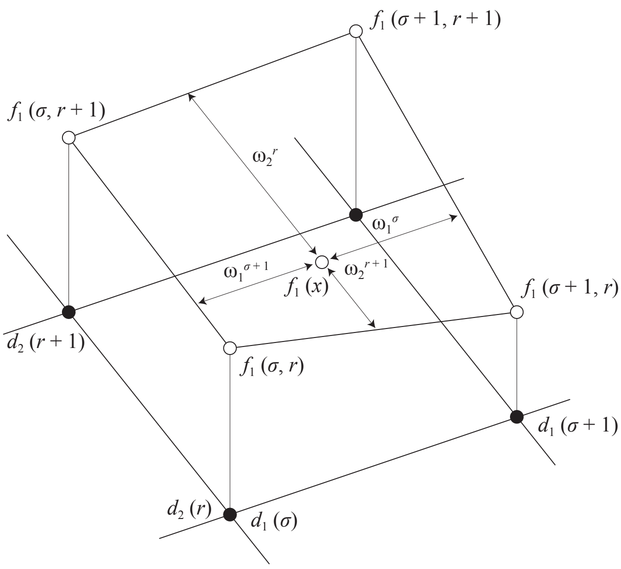

Figure 2: Piecewise region ($f_1(x)$, $x \in R_{\sigma\tau}$)

A key point in the system is that the state variable x is also expressed by a convex combination of $d(i, j)$ with respect to $w^i{}_1(x_1)$ and $w^j{}_2(x_2)$ just as in the case of $\dot{x}$. x is located inside $R\sigma\tau$ which is a rectangle: a hypercube in general. Thatis, the expression of x is polytopic with four vertices $d(i, j)$. The model of $\dot{x} = f(x)$ is built on a rectangle including x in the state space and it is also polytopic with four vertices $f(i, j)$.We call this form of the canonical model (2) parametric expression.

Representing $\dot{x}$ with x in Eq. (2), we can obtain the state space expression of the model which is found to be multilinear [Sugeno, 1999]. Therefore, the derived PML model has simple nonlinearity. In the case of the PL approximation, a PL model is built on simplexes partitioned in the state space, triangles in the two dimensional case. Note that any three points in the three dimensional space are spanned with an affine plane: $y = a + bx_1 + cx_2$. A PL model is continuous. It is, however, difficult to handle simplexes in the rectangular coordinate system.

Also we can see that any four points in the three dimensional space can be spanned with a bi-affine plane: $y = a + bx_1 + cx_2 + dx_1x_2$. In contract to a PL model, a PML model as such is built on rectangles with the four vertices $d(i, j)$, on hyper-cubes in a general dimensional space, partitioned in the state space; it well matches the rectangular coordinate system. Therefore, PML models would be applicable to control purpose.

2.2 Closed-loop systems

We consider a two-dimensional nonlinear control system.

$$\begin{cases} \dot{x} = f_o(x) + g_o(x)u(x), \\ y = h_o(x). \end{cases} \quad (3)$$

The PML model (4) can be constructed from the nonlinear system (3).

$$\begin{cases} \dot{x} = f(x) + g(x)u(x), \\ y = h(x), \end{cases} \quad (4)$$

where

$$\begin{cases} f(x) = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} w^i{}_1(x_1) w^j{}_2(x_2) f_o(i, j), \\ g(x) = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} w^i{}_1(x_1) w^j{}_2(x_2) g_o(i, j), \\ h(x) = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} w^i{}_1(x_1) w^j{}_2(x_2) h_o(i, j), \\ x = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} w^i{}_1(x_1) w^j{}_2(x_2) d(i, j), \end{cases} \quad (5)$$

and $f_o(i, j)$, $g_o(i, j)$, $h_o(i, j)$ and $d(i, j)$ are the vertices of the nonlinear system (3).

The modeling procedure in the region Ris as follows.

- Assign vertices $d(i, j)$ for $x_1 = d_1(\sigma)$, $d_1(\sigma + 1)$, $x_2 = d_2(\tau)$, $d_2(\tau + 1)$ of the state vector x, then the state space is partitioned into piecewise regions, see also Figure 2.

- Compute the vertices $f_o(i, j)$, $g_o(i, j)$ and $h_o(i, j)$ in equation (5), by substituting the values of $x_1 = d_1(\sigma)$, $d_1(\sigma + 1)$ and $x_2 = d_2(\tau)$, $d_2(\tau + 1)$ into original nonlinear functions $f_o(x)$, $g_o(x)$ and $h_o(x)$ in the system (3). Figure 2 illustrates the expression of $f(x)$ and $x \in R\sigma\tau$.

The overall PML model can be obtained automatically when all the vertices are assigned. Note that $f(x)$, $g(x)$ and $h(x)$ in the PML model coincide with those in the original system at the vertices of all the regions.

3. Design of LUT controllers for nonlinear systems with PB modeling and I/O linearization

This section deals with the I/O linearization of nonlinear control systems approximated with PML models. We consider, in particular, nonlinear systems of the second and third orders and show their I/O linearization based on PML models in detail. We also show that in the case of PML systems, the I/O linearization (the feedback-linearization in general) may be applicable to a global region by avoiding the restrictions of the conventional linearization of nonlinear control system: the restriction concerning the relative degree.

A nonlinear system is not always feedback-linearizable since the linearization conditions are not always satisfied. For instance, as is often the case, the relative degree is only defined in a restricted region. This problem is discussed in [Shankar, 1999]. There is even the case that the relative degree is not defined at the origin and, hence, the exact feedback linearization is not applicable. For such cases, various methods of approximate linearization have been suggested. As one of the methods, there is an idea to approximate a nonlinear system with a feedback-linearizable nonlinear system. The PML system is found to be easily linearizable and also, as is stated, it is a universal approximator for nonlinear systems. Therefore, the PML system can be also used in this context as a model of nonlinear systems for the feedback linearization.

4. Numerical example

Consider the following nonlinear system:

$$\begin{cases} \dot{x} = f_o(x) + g_o u(x) = \begin{pmatrix} \sin x_2 \\ -x^2{}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \\ y = h_o(x) = x_1, \end{cases} \tag{6}$$

Using the ordinary I/O feedback linearization, we design the linearizing controller for the nonlinear system (6)

$$u = \frac{-L^2{}_{fo}h_o(x) + v}{L_{go}L_{fo}h_o(x)} = -x^2{}_1 + \frac{1}{\cos x_2} v \tag{7}$$

The operation region of x_2 in (6) is considered in $[-\pi, \pi]$, but the controller (7) cannot be applied to stabilize the nonlinear system (6) in the outside of $-\pi / 2 < x_2 < \pi / 2$, since the relative degree is not defined at $x_2 = \pm\pi / 2$.

Now divide the state space of the nonlinear system (6) as $x_1 \in \{-2.0, -1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5, 2.0\}$ and $x_2 \in \{-7\pi / 6, -\pi / 4, 0, \pi / 4, 7\pi / 6\}$, then the PML model is constructed as

$$\begin{cases} \dot{x} = f(x) + gu(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \\ y = h(x) = x_1, \end{cases}$$

where

$$f_1(x) = \sum_{j=\tau}^{\tau+1} w^j{}_2(x_2) f_1(\cdot, j),\ f_2(x) = \sum_{i=\sigma}^{\sigma+1} w^i{}_1(x_1) f_2(i, \cdot),$$
$$f_1(\cdot, j) = f_1(p, j),\ f_2(i, \cdot) = f_2(i, q),$$
$$p = \sigma, \sigma + 1,\ q = \tau, \tau + 1.$$

The PML models of $f_1(x)$ and $f_2(x)$ are shown in Table 1. We design the LUT controller from $u(x)$ shown at (8) as is explained

$$u(x) = \frac{-L^2{}_f h(x)}{L_g L_f h(x)} + \frac{1}{L_g L_f h(x)} v \tag{8}$$

where

$$v = -Kz = -(1.31\ 0.938)\ z.$$

The LUT controller can be applied to the outside of the bound $-\pi / 2 < x_2 < \pi / 2$ because $L_g L_f h(x) \neq 0$. Table 2 shows the vertices of the LUT controller. The LUT controller can apply to stabilize the wider region than the I/O linearization controller (7). In fact, we obtain from the original system (6) that $L_g L_f h(x) = (\sin d_2(\tau$

Table 1: PB models of $f_1(x)$ and $f_2(x)$

$f_1(i, j)$	$d_2(-2)$	$d_2(-1)$	$d_2(0)$	$d_2(1)$	$d_2(2)$
$d_1(-4)$	−0.866	−0.707	0	0.707	0.866
$d_1(-3)$	−0.866	−0.707	0	0.707	0.866
$d_1(-2)$	−0.866	−0.707	0	0.707	0.866
$d_1(-1)$	−0.866	−0.707	0	0.707	0.866
$d_1(-0)$	−0.866	−0.707	0	0.707	0.866
$d_1(1)$	−0.866	−0.707	0	0.707	0.866
$d_1(2)$	−0.866	−0.707	0	0.707	0.866
$d_1(3)$	−0.866	−0.707	0	0.707	0.866
$d_1(4)$	−0.866	−0.707	0	0.707	0.866

$f_2(i, j)$	$d_2(-2)$	$d_2(-1)$	$d_2(0)$	$d_2(1)$	$d_2(2)$
$d_1(-4)$	−4.00	−4.00	−4.00	−4.00	−4.00
$d_1(-3)$	−2.25	−2.25	−2.25	−2.25	−2.25
$d_1(-2)$	−1.00	−1.00	−1.00	−1.00	−1.00
$d_1(-1)$	−0.25	−0.25	−0.25	−0.25	−0.25
$d_1(-0)$	0	0	0	0	0
$d_1(1)$	−0.25	−0.25	−0.25	−0.25	−0.25
$d_1(2)$	−1.00	−1.00	−1.00	−1.00	−1.00
$d_1(3)$	−2.25	−2.25	−2.25	−2.25	−2.25
$d_1(4)$	−4.00	−4.00	−4.00	−4.00	−4.00

Table 2: LUT controller of $u(i, j)$

$u_1(i, j)$	$d_2(-2)$	$d_2(-1)$	$d_2(0)$	$d_2(1)$	$d_2(2)$
$d_1(-4)$	41.7	31.6	6.91	6.09	9.44
$d_1(-3)$	34.6	24.5	4.43	3.61	2.29
$d_1(-2)$	27.9	17.8	2.45	1.64	–4.36
$d_1(-1)$	21.8	11.7	0.978	0.161	–10.5
$d_1(-0)$	16.1	6.06	0	–0.817	–16.1
$d_1(1)$	11.0	0.909	–0.478	–1.29	–21.3
$d_1(2)$	6.36	–3.74	–0.457	–1.27	–25.9
$d_1(3)$	2.20	–7.90	0.063	–0.754	–30.1
$d_1(4)$	–1.44	–11.5	1.08	0.266	–33.7

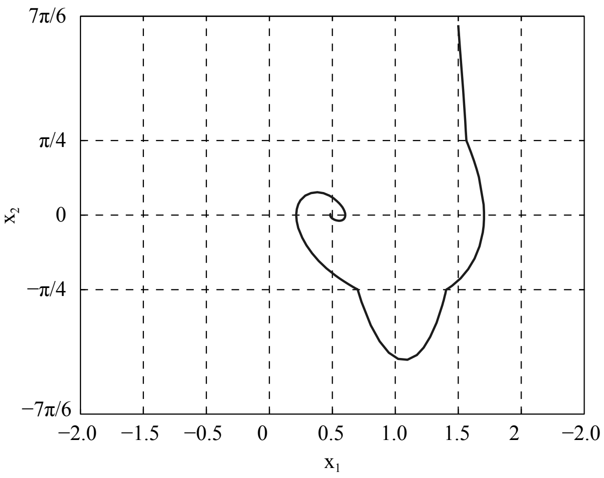

Figure 3: Simulation result of two-dimensional case

+ 1) – sin $d_2(\tau)$) / ($d_2(\tau + 1) - d_2(\tau)$) which is, in the current partition of the state space, not 0 for all $x_2 \in [-\pi, \pi]$, while in the I/O linearization of the original system, $L_gL_fh(x) = \cos x_2$. Figure 3 shows the simulation results in the initial condition $x(0) = (1, 2)^T$.

5. CONCLUSIONS

We have proposed the stabilization of nonlinear control systems approximated with PML models based on the inputoutput linearization. The designed controller is represented by the LUT. Also we have shown that by applying PML models to nonlinear control systems, we may drastically enlarge the feedback-linearizable region, since the relative degree is well defined.

Through the examples, we have shown that a combination of the PML modeling and the feedback-linearization could be a very powerful tool for the analysis and synthesis of nonlinear control systems. The PML system could be considered as a feedback-linearizable approximation model for nonlinear systems. An illustrative example has been given to show the validity of the proposed methods.

Now we are researching a tracking control for vehicle control systems. We are aiming to realize an unmanned vehicle control using our methods. We assume unmanned vehicle controls will become more important in our life, especially for overseas travelers. In most countries, right-hand traffic has been adopted as a traffic rule. So it might be difficult for overseas travelers to drive in Japan. Now overseas travelers are expected to increase, the unmanned technology will become all the more crucial for Japan.

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