A consideration on Buffon's needle problem

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Abstract
Although there are many delicate tasks performed by the fingertips, such as a famous doctor's scalpel treatment, calligrapher's brushwork, and craftsman's delicate work, there are few evaluation methods in those fields. In this study, evaluations are performed using "Buffon's needle problem," and the important factor is the random drop of a needle held by the fingertips. It is generally known that an approximate value of pi (circle ratio) can be found using Buffon's needle problem. Therefore, the approximation of pi by a needle dropped at random is calculated, and the closer the approximation is to true pi, the higher the accuracy of randomly dropping the needle held by the tip of a finger is judged to be. The purpose of this paper is to clarify the characteristics of "randomly dropping a needle held by the fingertips" using Buffon's needle problem. Computer simulations and actual experiments with a subject are performed in this study. And, the accuracy of randomly dropping the needle held at the fingertip is improved.

Key words
fingertip work, Buffon's needle problem, approximate value of pi, simulation, Monte Carlo method

1. Introduction

"Skillfulness" means a kind of dexterity and it is called "kochi-sei'" in Japanese and "kochi" means the elaborate and detailed appearance, and kochi-sei means a kind of dexterity of the fingers. Picking up objects with one's fingertips requires recognizing the object in front of one's eyes and focusing one's attention on the fingertips. Repeating this action will improve concentration (Study Hacker, 2018).

In this study, an evaluation on how to randomly drop a needle held by the fingertips using "Buffon's needle problem" is carried out.

The aim is to derive the features and hints that lead to improved accuracy by clarifying the characteristics of "how to randomly drop a needle held by one's fingertips".

In conclusion, a subject tried experimenting with the idea of dropping the needle vertically, but the accuracy of his fingertips did not improve. This is because the current position is unknown and the subject does not understand what condition it is in. Therefore, each time he dropped the needle, the experimenter calculated the predicted value of pi and tried to fine-tune the experimental process. The explanations, computer simulations, theoretical values and empirical experiments about Buffon's needle problem are explained in the following sections.

2. Buffon's needle problem

Buffon's needle problem is a mathematical problem posed by a French mathematician named Buffon. This problem was developed when an American flag was placed on a table and a needle with a length equal to the width of each stripe of the flag was randomly dropped onto the flag from above. And, the problem is to derive the probability of crossing the boundary between the stripes.

It is said that one of the reasons why this problem became famous is that it derives an approximation of pi (Ishida, 1959; Ishikawa, 1960).

3. Computer simulation

Firstly a computer simulation of Buffon's needle problem is considered (Kimura and Oyabu, 1989) and the following is assumed that the needle falls completely on the striped part of the flag. There is no need to distinguish between the left and right positions of the needles in the horizontal direction of the striped pattern as shown in Figure 1. And, only the center position y of the needle in the vertical direction of one stripe and the angle θ of the needle are important as shown in Figure 2.

It can be determined whether the needle crossed the border between the two stripes or not as follows using these

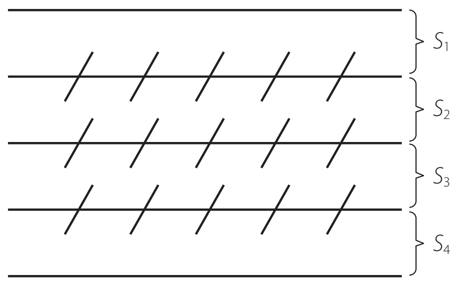


Figure 1: Relationship between needles and stripes
Note: *Si* = stripe; **/** = needle.

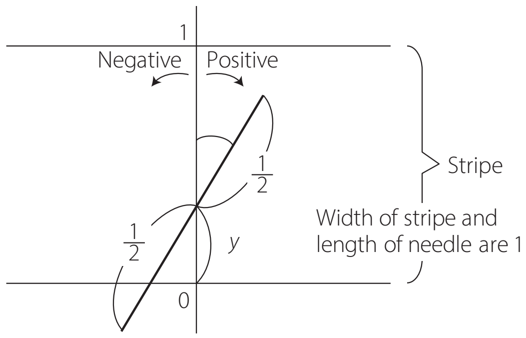


Figure 2: Relationship between the center position *y* of needle and needle angle *θ*

two elements (y and θ):

$$1 - y \leqq (1 / 2) \cdot \cos\theta \ \text{ when } \ 1 \geqq y \geqq 1 / 2 \quad (1)$$

$$y \leqq (1 / 2) \cdot \cos\theta \ \text{ when } \ 0 \leqq y < 1 / 2 \quad (2)$$

It is considered that the simulated needle crosses the boundary line as shown in Figure 3 when the above conditions are satisfied. There is no distinction between the tips of the needles and $-\pi / 2 \leqq \theta \leqq \pi / 2$ at this time.

3.1 Experiment 1

The probability that the needle crosses the boundary is derived when the number of times the needle falls is increased to 100, 1000, and 10000 times. The extended form rand() for uniformly distributed random numbers was used in this experiment.

3.2 Result 1

The flowchart of the simulation for Buffon's needle problem is shown in Figure 4 and the results are shown in Table 1.

Although RND(1) is generally used for uniformly distributed random numbers, the accuracy is not good, so the extended form rand() is used in this study.

Mersenne twister (Kobayashi, 2022) and Xorshift (Xorshift, 2022) are other methods. Mersenne twister has the highest accuracy, but the environment settings are difficult when it is used. The method is not used in the simulation experiment. The accuracy of Xorshift was almost the same as the extended rand() method. The method is a uniformly distributed random number as follows (Kobayashi, 2022):

Use the C language built-in function rand() and divide "rand()+0.5" by "RAND_MAX+1" to distribute it evenly without being biased toward either end.

The value of RAND_MAX is relatively small depending on the processing system, so the uniformly distributed random numbers were calculated using multiple rand() function values as follows:

```
double urand(){
    double m, a;
```

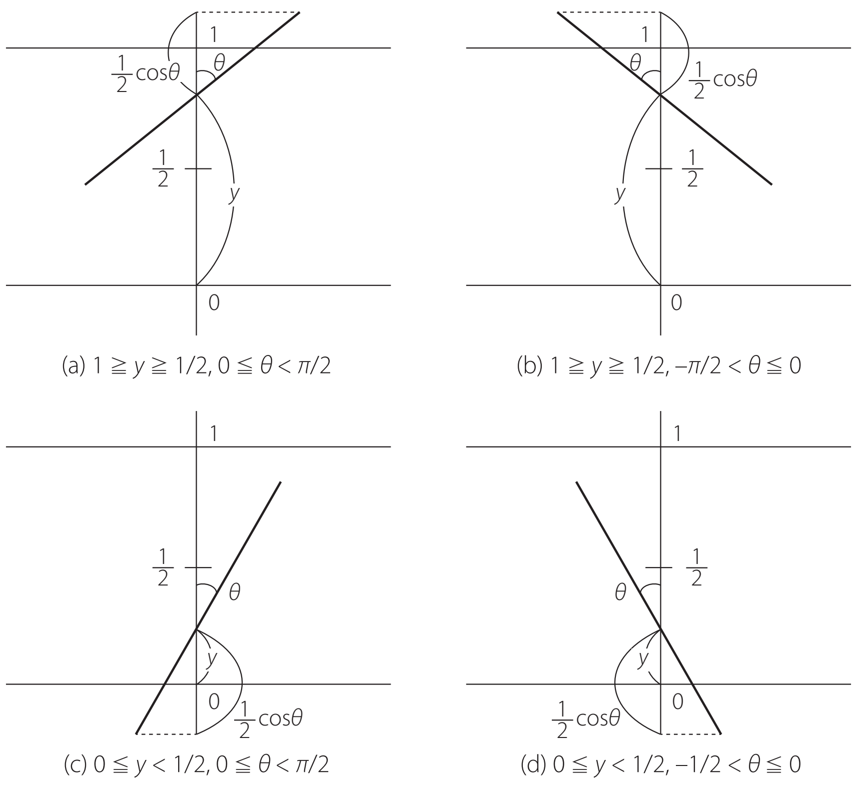


Figure 3: Illustrations when needle crosses boundary between stripes

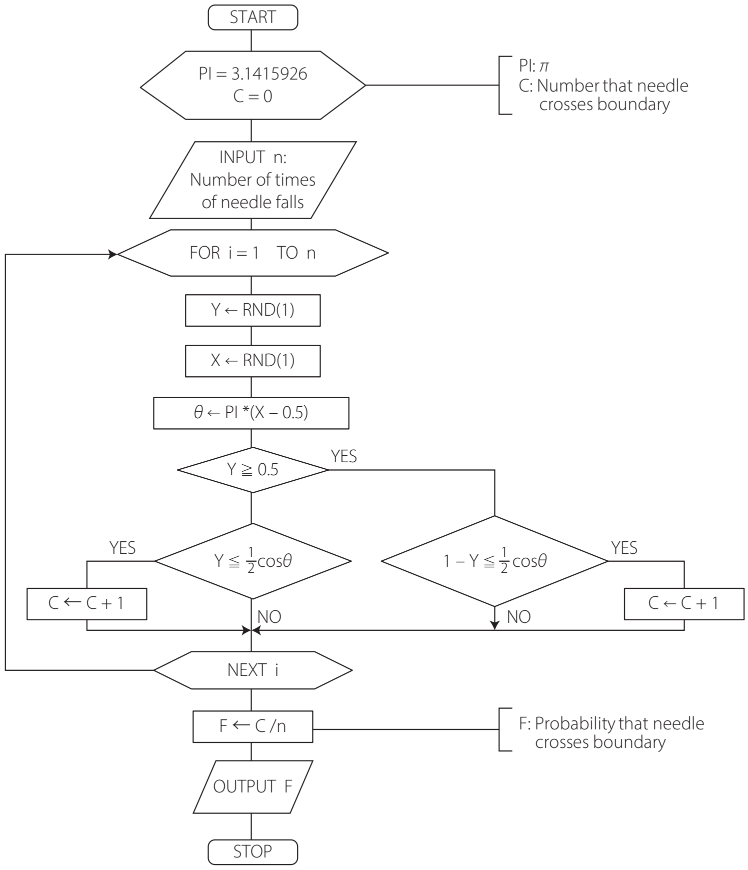


Figure 4: Flowchart for Buffon's needle problem

Table 1: Results of Experiment 1

Number of needle drops	Probability of crossing boundary of stripes	Approximate value of pi
100	0.6900	2.8985507
1000	0.6290	3.1796502
10000	0.6371	3.1392246

```
  m=RAND_MAX+1;
  a=(rand()+0.5)/m;
  a=(rand()+a)/m;
  return (rand()+a)/m;
}
```

4. Theoretical value for Buffon's needle problem

Firstly, the theoretical value for Buffon's needle problem is derived (Nakamura, 2017; Mathtrain, 2020). The limit for crossing the straight line of the boundary of the striped pattern is when the tip of the needle touches the straight line of the border, as shown in Figure 5. The relational expression between θ and y in this case is as follows:

$$\cos\theta = (1 - y) / 0.5 \quad (3)$$

$$y = 1 - 0.5 \cdot \cos\theta \quad (4)$$

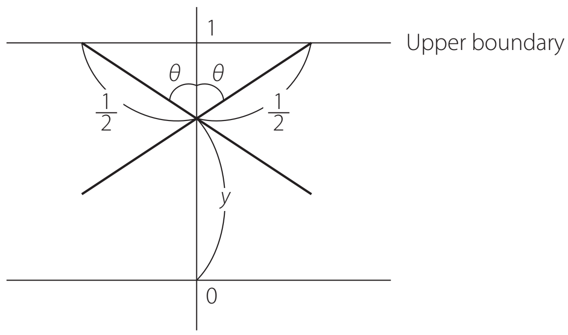


Figure 5: Limit at crossing upper straight line (upper boundary)

It crosses the boundary line when θ and y satisfy

$$y \geqq 1 - 0.5 \cdot \cos\theta \quad (5)$$

The limit for crossing the straight line of the lower boundary is also when the tip of the needle touches the straight line of the boundary, as shown in Figure 6. The relational expression is as follows:

$$\cos\theta = y / 0.5 \quad (6)$$

$$y = 0.5 \cdot \cos\theta \quad (7)$$

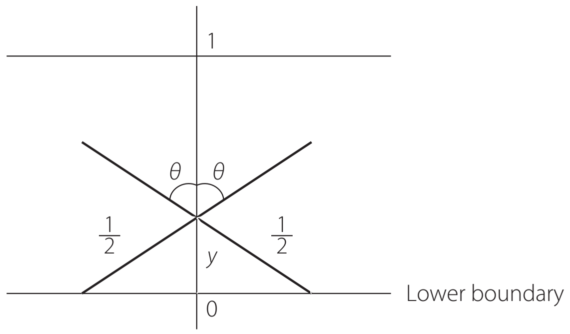


Figure 6: Limit at crossing lower straight line (lower boundary)

When θ and y satisfy $y \leqq 0.5 \cdot \cos\theta$, it crosses the boundary straight line. The relationship between θ and y when crossing the boundary is shown in Figure 7.

The probability F of crossing the boundary can be calculated using the following formula from the above:

F = Area of shaded area (S) / Area of entire rectangle (V)

In the above formula,

$$V = \pi \tag{8}$$

$$\begin{aligned} S &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0.5\cdot\cos\theta d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta \\ &= [\sin\theta]_{-\pi/2}^{\pi/2} \\ &= 1-(-1) \\ &= 2 \end{aligned} \tag{9}$$

Therefore, the value of F becomes as follows:

$$\begin{aligned} F &= S / V \\ &= 2 / \pi \end{aligned} \tag{10}$$

Due to the above result, the value of π is $2 / F$ and an approximation of π can be derived from the simulation of Buffon's needle problem.

5. Demonstration experiments

The following demonstration experiments were performed in different environments and the characteristics were revealed. The first experiment examines how to randomly drop a needle held by the fingertips. And, the features and hints that lead to improved accuracy can be grasped by using the results. Firstly the stripes of the American flag are traced onto paper as illustrated in Figure 1 and the paper is put on a desk. After that, Buffon's needle problem is retested. The following reports were received by Analytics (2023). One report is that Captain Fox obtained an approximation of 3.1595 for pi after 1,030 trials in a follow-up to Buffon's needle problem in 1864. Another is that Wolf came up with an approximate value of 3.1596 after 5,000 trials. From the above, it can be seen that it may not be possible to expect an improvement in the accuracy of the approximate value even after exceeding 1,000 trials or more. Therefore, the maximum number of repetitions per person was set at 1,000 times in the additional trials of this experiment, in which the needles held by the fingertips were randomly dropped.

5.1 Experiment 2

A subject was asked to repeat Buffon's needle problem without any conditions and the total number of trials was set to 1000. And, an approximate value of pi was calculated from the "probability of crossing the boundary between stripes" and then graphed. The values of pi were derived for every 1 to 50 trials, 1 to 100 trials, 1 to 200 trials, . . ., and 1 to 1,000 times. The subject sat in a chair and performed the trials on a desk. The length of the needle was 39 mm and striped lines drawn on A4 size cardboard. The needle was dropped from a height of 30 cm.

5.2 Result 2

The experimental results are shown in Figure 8. The approximate value of pi converges around 600 repeated trials however the relative error is large (34 %). Sometimes the needle would fall off the cardboard or stick out. Although the subject

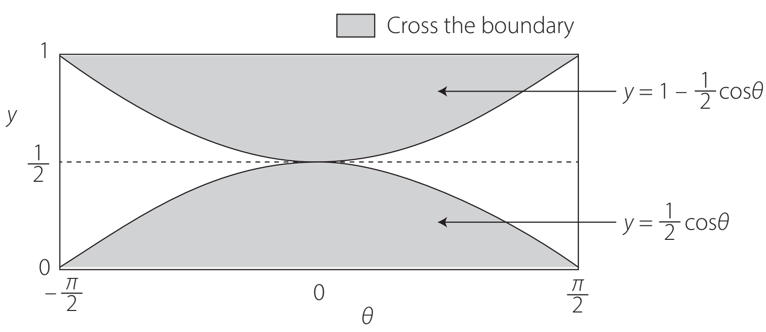


Figure 7: Relationship between θ and y when crossing boundary

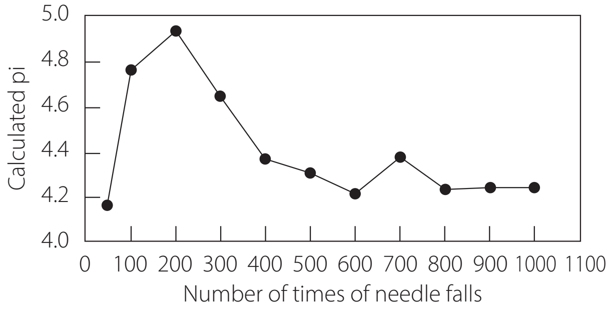


Figure 8: Relationship between number of needle falls and pi (Result of Experiment 2)

thinks he is conducting the trial experiments, some factors influence the reality, in which the needle drops randomly. For this reason, the length of the needle was shortened and the cardboard on which the needle was dropped was changed to B4 size.

5.3 Experiment 3

It was investigated whether the needle length affected the accuracy improvement when the needle drop height was constant. The height of dropping the needle was 30 cm, and the length of the needle was 29 mm. At this time, Buffon's needle problem was tried by throwing the needle 50, 100, 200, . . ., 1,000 times. However, the calculations were made by adding 100 times to the previous number for experiments after 200 times. Only one subject was adopted and the cardboard was B4 size.

5.4 Result 3

The experimental result is expressed in Figure 9. It seemed to converge once after about 700 trials, but after that it diverged. It is thought that this was caused by the influence of the object in front of the subject, a lack of awareness of his fingertips, and a lack of concentration.

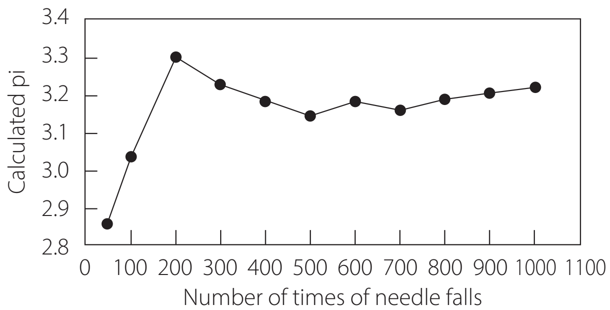


Figure 9: Relationship between number of needle falls and pi (Result of Experiment 3)

5.5 Experiment 4

It was investigated whether the way the subject holds the needle and the position he holds it affects the accuracy of the derived pi. The experiments were conducted with the following four patterns: the subject holds the needle with his thumb and index finger and also holding with thumb and little finger (two patterns), and there are two positions to hold the needle—at the middle of the needle and at the end of the needle (two patterns). There was only one subject and the number of times the needle was thrown was 100, 200, . . ., 1,000 times, similar to Experiment 2. However, it was calculated by adding 100 times to the previous time.

5.6 Result 4

The experimental results are shown in Figure 10. The subject felt that the needle is easier to hold and control at the middle than at the tip. On the other hand, it is likely to fluctuate and be difficult to control when holding the tip of the needle. And, it was demonstrated that the index finger is easier to control than the little finger.

The reason why accuracy could not be improved when holding the needle between the thumb and index finger, which should be easier to control, is important. It is thought that this is because there are more influential factors. It can also be thought that it depends on how vertically the needle is dropped. Therefore, the subject tried to drop the needle vertically in the next experiment.

5.7 Experiment 5

The experiment was performed by speaking out loud and trying to drop the needle vertically. The needle was held at the end with the thumb and forefinger. Buffon's needle problem was performed by releasing the needle 100, 200, . . ., 1,000 times. However, it was calculated by adding 100 times to the previous time. There was only one subject, similar to the previous experiments.

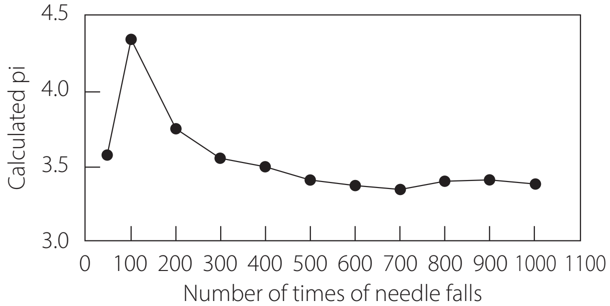
5.8 Result 5

The experimental result is shown in Figure 11. The subject experimented with trying to drop the needle vertically, but the accuracy of his fingertips did not improve. The reason for this is that the drop location is unknown and this is because the subject does not know what condition it is in.

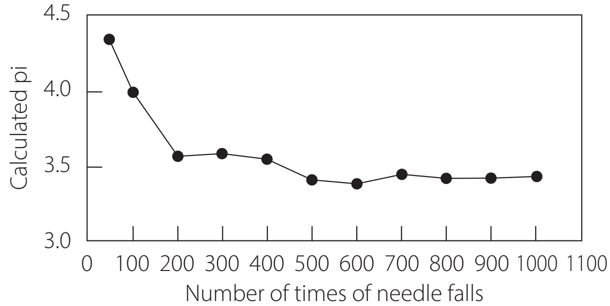
Therefore, he decided to calculate the predicted value of pi every time he dropped the needle and tried to make fine adjustments. Specifically, it is the same as Experiment 6 below.

5.9 Experiment 6

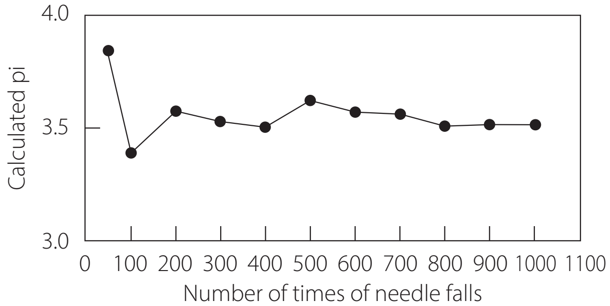
If the predicted value of pi is larger than the actual value, the subject holds the end of the needle between his thumb and index finger and drops it from just above the boundary between the stripes in order to increase the probability F of crossing the boundary of the stripes. And, if the predicted value of pi is smaller than the actual value, the subject holds



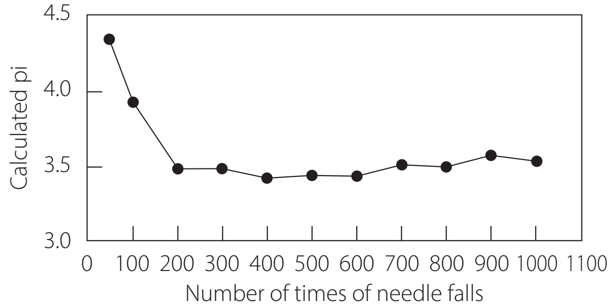
(a) Holding needle in middle between thumb and index finger



(b) Holding rear end of needle with thumb and index finger



(c) Holding needle in middle between thumb and little finger



(d) Holding rear end of needle with thumb and little finger

Figure 10: Results of Experiment 4

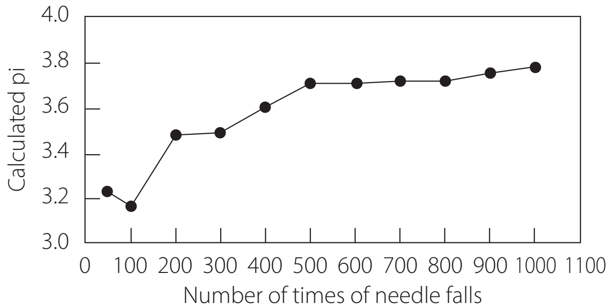


Figure 11: Holding end of needle with thumb and forefinger (conscious of verticality) (Result of Experiment 5)

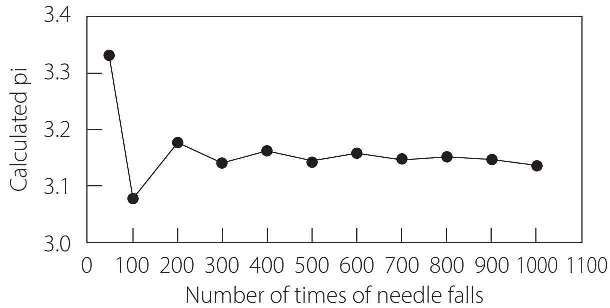


Figure 12: Holding rear end of needle with thumb and forefinger (Result of Experiment 6)

the end of the needle between his thumb and forefinger and drops it into the stripe to make F smaller. He gradually reduces the adjustment amount of the needle drop position. In this way, the subject performed Buffon's needle problem by releasing the needle 100, 200, . . ., 1000 times. However, it was calculated by adding 100 times to the previous time. There was only one subject.

5.10 Result 6

The experimental result is shown in Figure 12. The value of F was adjusted by changing the location where the needle was dropped, either directly above the stripes or between the stripes, depending on the calculated pi. This achieved high precision.

6. Conclusion

Many people have experimented with Buffon's needle problem without using a computer. They simply continued to throw needles using paper. Table 2 shows the top five throwers with the most number of times in history (Analytics, 2023). Wolf, who conducted 5,000 experiments, took first place and there is a huge difference between him and the second place. The derived pi is 3.1596 (the exact value is 3.1415926 . . .). Its accuracy ranks second. Even so, Lazzarini's pi is also extremely good.

The above are the results of experiments conducted in the past, however the results of the experiments conducted in this present study are summarized below.

Although the fingertip accuracy improved in Experiment 2 when the needle was dropped at a constant height, an improvement in fingertip accuracy could not be confirmed in other experiments. And, it was also found that the accuracy of fingertips deteriorated as the number of trials increased.

In Experiment 5, the subject experimented with trying to

Table 2: Top five throwers with the most number of times in Buffon needle problem experiments

Rank	Name	Year	Number of times dropped	Derived pi
1st	Wolf	1800s	5000	3.1596
2nd	Lazzarini	1901	3408	3.1415929
3rd	Smith Daveldeen	1855	3204	3.1553
4th	Reina	1925	2520	3.1795
5th	Captain Fox	1864	1030	3.1595

drop the needle vertically, however, the accuracy of his fingertips did not improve. This is because the current situation (position) is unknown and the subject does not know what state it is in.

Therefore, the subject decided to calculate the predicted value of pi every time he dropped the needle and tried to make fine adjustments. This was shown specifically in Experiment 6. This made it possible to improve accuracy.

References

Analytics (2023). Buffon's needle experiment: Find Pi by just throwing a needle (Retrieved June 14, 2023 from https://analytics-notty.tech/buffon-needle-experiment/trackback/). (in Japanese)

Ishida, A. (1959). Buffon's needle problem. *Science of Machine*, Vol. 11, No. 12, 1592-1596.

Ishikawa, E. (1960). About Buffon's needle problem. *The Annual Report of the Department of Liberal Arts, the Iwate University*, Vol. 17, No. 2, 9-11.

Kimura, H. and Oyabu, T. (1989). *Introduction to information science*. Kyoritsu Shuppan. (in Japanese)

Kobayashi, K. (2022). Generation program of Pseudo-random number (Retrieved June 14, 2023 from http://kobayashi.hub.hit-u.ac.jp/topics/rand.html). (in Japanese)

Mathtrain (2020). Buffon's needle problem and derivation of probability (Retrieved June 14, 2023 from https://mathtrain.jp/buffon). (in Japanese)

Nakamura, R. (2017). A world where pi appears: Buffon's needle problem (Retrieved June 14, 2023 from https://www.nli-research.co.jp/report/detail/id=56991?site=nli). (in Japanese)

Study Hacker (2018). What does "skillfulness" mean? I will teach you easy training methods and toys (Retrieved June 14, 2023 from https://kodomo-manabi-labo.net/skillfulness-in- fingers-and-hands). (in Japanese)

Xorshift (2022). Generation program of Pseudo-random number (Retrieved June 14, 2023 from https://kobayashi.hub.hit-u.ac.jp/topics/rand.html). (in Japanese)

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